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1986 J. Phys. A: Math. Gen. 19 L275

(http://iopscience.iop.org/0305-4470/19/5/011)

J. Phys. A: Math. Gen. 19 (1986) L275-L279. Printed in Great Britain

LETTER TO THE EDITOR

A transfer matrix study of conductivity and permeability exponents in continuum percolation

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Received 20 January 1986

Abstract. The conductivity and permeability exponents for the Swiss cheese model of continuum percolation are calculated numerically, using the transfer matrix method. In two dimensions, we find the conductivity exponent, t, to equal its universal value $t_{un} = 1.24$, while the permeability exponent $e = 2.53$ is considerably larger. The same exponents in three dimensions are also determined and found to be $t = 2.46$ and $e = 4.1$, greatly exceeding the universal value $t_{un} = 1.95$. The results are in fair agreement with earlier theoretical predictions.

The problem of non-universality of transport properties in percolating systems has been considered by several groups (Kogut and Straley 1979, Ben Mizrahi and Bergman 1981, Straley 1982). The model treated is that of conduction on a percolating lattice where the conductance of the bonds has the probability distribution

$$P(g) = \begin{cases} (1-p)\delta(g) + p(1-\alpha)g^{-\alpha} & g < 1 \\ 0 & g \geqslant 1 \end{cases} \tag{1}$$

with $\alpha < 1$.

For $\alpha > 0$, this distribution contains a large weight of poor conductors. Using such a distribution and specific model systems, Kogut and Straley (1979) calculated the critical exponent t defined by $\Sigma \propto (p - p_c)^t$ where Σ is the macroscopic conductivity. They concluded that $t(\alpha) = t_{un} + \alpha/(1-\alpha)$ where t_{un} is the exponent one would obtain on the same systems with distributions that satisfy $P(g) \to 0$ as $g \to 0$ for the conducting bonds. Ben-Mizrahi and Bergman (1981) calculated the same exponent using a Migdal-Kadanoff renormalisation group transformation and obtained $t(\alpha) = A + B/(1-\alpha)$, with dimension-dependent constants, A and B. Straley (1982) used the renormalised Skal-Shklovskii and de Gennes model, and obtained $t(\alpha) = (d-2)\nu + 1/(1-\alpha)$ where d is the dimensionality of the system and ν is the correlation length critical exponent.

Recently, Halperin *et al* (1985) considered the 'Swiss cheese model' (SCM) in which spherical (circular) holes of radius a are randomly placed in a 3D (2D) uniform medium. This model is mapped onto a discrete network of bonds where each bond represents a narrow region of width δ persisting over a length $l \propto \sqrt{a\delta}$. The conductivity of the network can then be calculated as in a lattice model, with the modification that the conductance of the bond is given by $g(\delta) \propto \delta^m$, where $m = \frac{3}{2}$ ($\frac{1}{2}$) in 3D (2D). Fluid flow in the space between spherical grains may also be described by this model. In this case, the permeability of each bond is $k(\delta) \propto \delta^m$ where $\delta = \frac{7}{2}$ ($\frac{5}{2}$) in 3D (2D).

0305-4470/86/050275+05$02.50

Since δ has a distribution which is finite at $\delta \to 0$ one obtains a distribution of the type given in (1) both for g and k, with $\alpha = 1 - 1/m$. This leads to different critical exponents t and e for the macroscopic conductivity and permeability, respectively, in contrast to the case of lattice percolation where $t = e = t_{un}$.

Using the nodes and links picture of the percolating backbone, the above authors obtained estimates for t and e. These estimates are identical to the expression obtained by Straley (1982) with α identified as $1 - 1/m$. Their results show that the exponents indeed differ considerably from the lattice exponent, t_{un}, except for the case of conductivity in 2D where $t = t_{un}$. Some of these predictions were confirmed numerically in two dimensions (Sen *et al* 1985).

In this letter we report on simulations to calculate the conductivity and permeability exponents for the SCM both in two and three dimensions. Although the SCM maps onto a random network we use a square or cubic lattice with a uniform coordination number (6 in 3D and 4 in 2D) and with a random occupation of bonds. We believe that this simplification will not affect the critical behaviour. Our results are summarised in table 1.

Table 1. Conductivity and permeability exponents in continuum (t, e) and lattice (t_{un}, e_{un}) percolation (our error bars are only the statistical errors). The earlier theoretical predictions are also presented.

	Our simulations			Earlier predictions	
	$t_{un} = e_{un}$	t	e	t	e
2D	1.24 ± 0.05	1.32 ± 0.05	2.53 ± 0.05	t_{un}	2.74[a] 2.5[b] 2.5[c]
3D	1.95 ± 0.1	2.46 ± 0.1	4.1 ± 0.3	2.45[a] 2.72[b] 2.5[c]	4.45[a] 4.72[b] 4.5[c]

[a] Kogut and Straley (1979)—we substituted our calculated t_{un} to evaluate their expression.
[b] Ben-Mizrahi and Bergman (1981).
[c] Straley (1982), Halperin *et al* (1985).

To obtain these results, we used the transfer matrix method (Derrida and Vannimenus 1982, Derrida *et al* 1983). In this method one starts with a layer containing L_x by L_y bonds (conductors) in the x and y directions, respectively. One then calculates the conductance matrix A_{ij} given by $I_i = \Sigma_j A_{ij} V_j$, where I_i, V_i are respectively the current and potential at site i of the layer. The next layer is constructed by adding vertical bonds at the sites i and connecting them by horizontal bonds. The new matrix A_{ij} is then calculated from the old one and from the conductances of the added bonds. A long strip is thus built up by repeating this procedure N times, $N \gg L_x$, L_y. The conductance per unit length, G/N, of the strip between two equipotential plates parallel to the xz plane can be calculated. The process is stopped when G/N saturates at a limiting value.

We set $L_x = L_y = L$ for the 3D calculations and $L_x = 0$, $L_y = L$ for the 2D ones. Finite-size scaling suggests that $G/N \propto L^{-t/\nu}$ in 3D and $G/N \propto L^{-(t/\nu+1)}$ in 2D, where ν is the correlation length exponent for percolation.

This method which was originally applied to the conductivity problem can be applied with no modifications to the calculation of fluid flow with the permeability

replacing conductivity. This follows from the analogy between the equations governing both phenomena. For the sake of concreteness we use the terminology appropriate to electrical conductivity but the discussion applies to fluid flow as well.

We have constructed the strips using bonds having conductances which are randomly chosen from a probability distribution given by (1) with $p = p_c$, the geometrical bond percolation threshold. We used $p_c = 0.5$ for 2D and $p_c = 0.2492$ (Wilke 1983) for 3D (however, see also Grassberger (1986)).

In two dimensions we performed our calculations with $m = 0$ (corresponding to $\alpha = -\infty$), $m = \frac{1}{2}$ ($\alpha = -1$) and $m = \frac{5}{2}$ ($\alpha = \frac{3}{5}$). These, respectively, correspond to lattice conductance ($p(g) = (1-p)\delta(g) + p\delta(1-g)$), continuum conductance and fluid flow in the SCM. One does not expect any difference in the critical exponents of the first two cases as the probability distribution of g vanishes at $g = 0$. The calculations were carried out for eight values of L, $5 \leqslant L \leqslant 40$, and for N up to 100 000. In each case, the limiting value of G/N was obtained to within 1% already at $N = 50\,000$. The results are presented in figure 1. We see that for $L \geqslant 10$ the points lie on straight lines. The slopes of the lines with $m = 0$ and $m = \frac{1}{2}$ are indeed approximately the same, while for $m = \frac{5}{2}$ the slope is significantly larger. The slopes were calculated by a least-squares fit of the data and we obtained $t/\nu = 0.93$, 1.00 and 1.88 for the three cases. To estimate the errors in these values, we calculated the slopes using different ranges of L. We believe the statistical errors to be approximately ± 0.05. Taking $\nu = \frac{4}{3}$ we get the results presented in table 1.

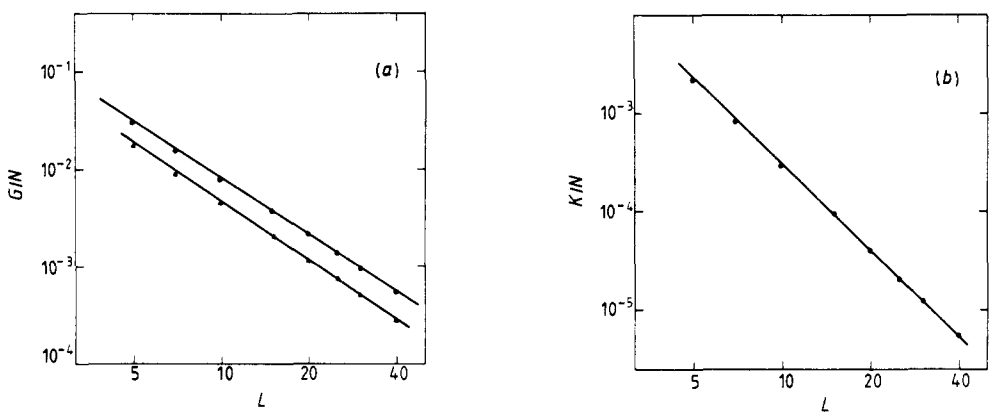

Figure 1. (*a*) Conductivity per unit length of the strip, G/N, as a function of the width of the strip, L, for two dimensions: (●), lattice percolation; (▲), continuum percolation. (*b*) Permeability per unit length of the strip K/N.

Similar calculations were performed in three dimensions. The cases investigated were the 3D analogues of those treated in 2D, namely $m = 0$ ($\alpha = -\infty$), $m = \frac{3}{2}$ ($\alpha = \frac{1}{3}$) and $m = \frac{7}{2}$ ($\alpha = \frac{5}{7}$). The strips were of size $L \times L \times N$ with $L = 3, 4, \ldots, 14$ and $N = 100\,000$. Again G/N saturates at smaller values of N. The results are presented in figure 2. The same kind of analysis yields $t/\nu = 2.2 \pm 0.1$, 2.8 ± 0.1, 4.7 ± 0.3. The exponents were obtained assuming $\nu = 0.88$ (Heermann and Stauffer 1981) and are presented in table 1.

We now discuss our results and compare them with earlier predictions. In 2D, we found that the conductivity exponent differs only by 0.08 from its universal value which we calculated by the same method. Within our numerical accuracy, this result is in

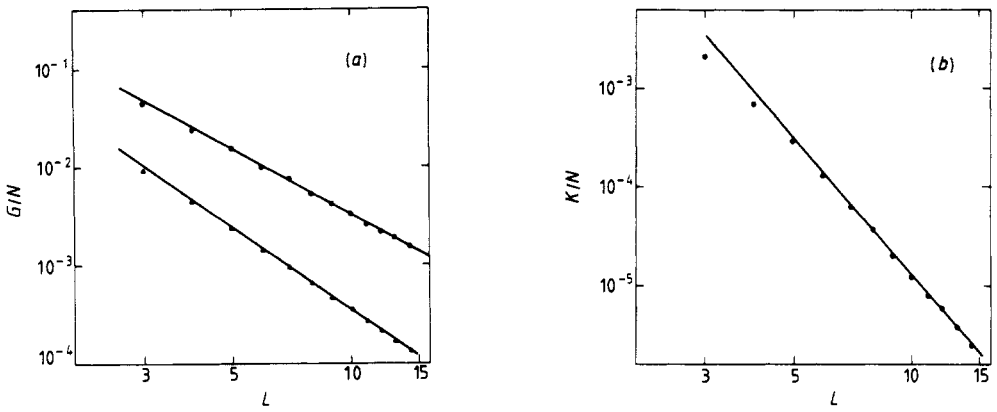

Figure 2. Same as figure 1, but in three dimensions.

agreement with the theoretical prediction that $t = t_{un}$ for $m < 1$. The permeability exponent, however, is found to differ considerably from t_{un}. Our result, $e = 2.53$, is in agreement with the lower bound $e = \frac{5}{2}$ given by Halperin *et al* (1985) and with the prediction of Ben-Mizrahi and Bergman (1981), $e = 2.50$, obtained by a Migdal-Kadanoff renormalisation group calculation. For comparison with the result of Kogut and Straley we substitute our t_{un} in their expression and obtain 2.74, which is somewhat higher than our result.

We note that our results in 2D agree quite well with those of Sen *et al* (1985) for the same model. These authors calculated the transport exponents by solving Kirchoff's equations on small $L \times L$ lattices using a matrix inversion procedure. This method has the inherent drawback that the results depend to some extent on the method used for averaging the conductivity over different realisations of the random network. This problem is avoided in the transfer matrix method since a sufficiently long random strip of width L always leads to a unique result for the conductivity.

In 3D both t and e are considerably larger than the corresponding t_{un} (which incidentally is identical to the t_{un} obtained by Derrida *et al* (1983) using the same method). The conductivity exponent we obtained, $t = 2.46 \pm 0.1$, compares well with the theoretical predictions, with the exception of that of Ben-Mizrahi and Bergman ($t = 2.72$). The latter prediction is probably less reliable than others, because the method used (Migdal-Kadanoff renormalisation group) becomes less accurate the higher the dimensionality. The permeability exponent ($e = 4.1 \pm 0.3$) is found to be smaller than all the predictions ($e = 4.5$-4.7). However, taking into account corrections to scaling, visible in our figures as a slight curvature on the log-log plots, our exponents would be increased somewhat, probably by not more than 0.1.

In summary, we have calculated the conductivity and fluid permeability exponents in a model of a continuum percolating system. We found deviations from the lattice values of these exponents. These deviations are similar to those predicted by theoretical considerations.

This letter was supported in part by grants from the Israel Academy of Sciences and Humanities, the US-Israel Binational Science Foundation and the Israel AEC. One of us (SM) also acknowledges a grant from the Israel Center for Absorption in Science.

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